Suggested Solutions to:
Regular Exam, Spring 2017
Industrial Organization
June 1, 2017

This version: June 20, 2017

Question 1: A vertical relationship and downstream duopoly

To the external examiner: The students had not seen this exact model before. But the model is of course based on material that they have seen in the course.

Part (a)

Given $d = 0$, the inverse demand functions can be written as $p_1 = 1 - q_1$ and $p_2 = 1 - q_2$, and each one of the two retailers is a monopoly firm in its market. We can solve for the (subgame perfect) equilibrium by using backward induction. Thus, firm D_1 chooses q_1 so as to maximize $\pi_1 = (1 - w - q_1)\, q_1$. Standard calculations yield the optimal quantity

$$\widehat{q}_1 = \max\left\{\frac{1-w}{2}, 0\right\}. \qquad (1)$$

The optimal quantity for firm D_2 is the same, $\widehat{q}_2 = \widehat{q}_1$. The upstream firm U, when choosing w, anticipates the optimal behavior of the downstream firms and thus maximizes the profit

$$\pi_U = (\widehat{q}_1 + \widehat{q}_2)\, w = 2 \max\left\{\frac{1-w}{2}, 0\right\} w.$$

Choosing any $w \geq 1$ would yield zero profit, which cannot be optimal, so the relevant profit expression can be written as $\pi_U = (1 - w)\, w$, which is maximized at $w^* = \frac{1}{2}$. This in turn yields (by using (1)) the equilibrium output levels $q_1^* = q_2^* = \frac{1-\frac{1}{2}}{2} = \frac{1}{4}$. Moreover, the equilibrium prices are obtained by plugging these quantities into the demand functions:

$$p_1^* = p_2^* = 1 - \frac{1}{4} = \frac{3}{4}.$$

Part (b)

As in part (a), we can solve the game by backward induction. At the second stage, firm D_2 chooses q_2 so as to maximize $\pi_2 = (1 - w - dq_1 - q_2)\, q_2$. Standard calculations yield the best response

$$q_2 = \max\left\{\frac{1-w-dq_1}{2}, 0\right\}. \qquad (2)$$

The integrated firm $\widehat{U}$ chooses q_1 so as to maximize $\pi_{\widehat{U}} = (1 - q_1 - dq_2)\, q_1 + wq_2$. Standard calculations yield the best response

$$q_1 = \max\left\{\frac{1-dq_2}{2}, 0\right\}. \qquad (3)$$

Given that the wholesale price that D_2 must pay is, at the first stage, chosen by $\widehat{U}$, and that D_2 also is $\widehat{U}$'s second stage rival, we may conjecture that there is an equilibrium of the overall game in which $\widetilde{q}_2 = 0$. We should therefore consider this possibility. If $\widetilde{q}_2 = 0$, then (by (3)) $\widetilde{q}_1 = \frac{1}{2}$; this means, by (2), that a zero output by D_2 is a best response if $\frac{1-w-d\frac{1}{2}}{2} \leq 0$ or $w \geq \frac{2-d}{2}$.[1]

We should also consider the possibility that both firms choose a positive output at stage 2. It follows from (2) and (3) that the equilibrium quantities are then characterized by the following equation system (here on matrix form):

$$\begin{bmatrix} 2 & d \\ d & 2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1-w \end{bmatrix}.$$

Cramer's rule yields the following equilibrium quantities (other ways of solving the equation system are

[1] We can rule out the (counter-intuitive) possibility that there is an equilibrium of the overall game in which $q_1 = 0$. If $q_1 = 0$, then (by (2)) $q_2 = \frac{1-w}{2}$; this means, by (3), that a zero output by D_1 is a best response if $\frac{1-d\frac{1-w}{2}}{2} \leq 0$, which cannot hold.

of course also fine):

$$\widehat{q}_1 = \frac{2 - d\,(1-w)}{4-d^2}, \tag{4}$$

$$\widehat{q}_2 = \frac{2\,(1-w) - d}{4-d^2}. \tag{5}$$

Note that D_2 indeed chooses a positive quantity if, and only if, $\widehat{q}_2 \geq 0$ or $w \leq \frac{2-d}{2}$.

At the first stage, $\widehat{U}$ chooses w so as to maximize the following profit expression:

$$\widehat{\pi}_{\widehat{U}} = \begin{cases} \left(1 - \widehat{q}_1 - d\widehat{q}_2\right)\widehat{q}_1 + w\widehat{q}_2 & \text{if } w \leq \frac{2-d}{2} \\ \left(1 - \frac{1}{2}\right)\frac{1}{2} & \text{if } w \geq \frac{2-d}{2}. \end{cases}$$

That is, any $w \geq \frac{2-d}{2}$ yields the profit $\frac{1}{4}$. If $w \leq \frac{2-d}{2}$, so that the first line of the profit expression is relevant, the following first-order condition must be satisfied:[2]

$$\begin{aligned} \frac{\partial \widehat{\pi}_{\widehat{U}}}{\partial w} &= -d\widehat{q}_1 \frac{\partial \widehat{q}_2}{\partial w} + \widehat{q}_2 + w\frac{\partial \widehat{q}_2}{\partial w} \\ &= \widehat{q}_2 + \left(w - d\widehat{q}_1\right)\frac{\partial \widehat{q}_2}{\partial w} \\ &= \frac{2\,(1-w) - d}{4-d^2} - \frac{2\left(w - d\widehat{q}_1\right)}{4-d^2} \\ &= \frac{2\,(1-2w) - d}{4-d^2} + \frac{2d}{4-d^2}\widehat{q}_1 \\ &= \frac{2\,(1-2w) - d}{4-d^2} + \frac{2d\left[2 - d\,(1-w)\right]}{\left(4-d^2\right)^2} \\ &= 0. \end{aligned}$$

Solving for w yields

$$\begin{aligned} &\left[2\,(1-2w) - d\right]\left(4-d^2\right) = -2d\left[2 - d\,(1-w)\right] \Leftrightarrow \\ &\qquad w^* = \frac{(2-d)\left(4-d^2\right) + 2d\,(2-d)}{-2d^2 + 4\left(4-d^2\right)} \\ &\qquad\quad = \frac{(2-d)\left(4+2d-d^2\right)}{2\left(8-3d^2\right)}. \end{aligned} \tag{6}$$

For later reference, also note that

$$\begin{aligned} & 1 - w^* \\ = \; & \frac{2\left(8-3d^2\right) - (2-d)\left(4+2d-d^2\right)}{2\left(8-3d^2\right)} \\ = \; & \frac{16 - 6d^2 - \left(8+4d-2d^2\right) + \left(4d+2d^2-d^3\right)}{2\left(8-3d^2\right)} \\ = \; & \frac{8-2d^2-d^3}{2\left(8-3d^2\right)}. \end{aligned} \tag{7}$$

Note that we have $w^* \leq \frac{2-d}{2}$ for all $d \leq 1$. This means that w^* is indeed the overall optimum and that D_2 will produce a positive quantity for all $d < 1$.

We can now obtain the equilibrium outputs by plugging (7) into (4) and (5). Doing that yields

$$\begin{aligned} q_1^* &= \frac{2 - d\,(1-w^*)}{4-d^2} = \frac{2 - d\left(\frac{8-2d^2-d^3}{2(8-3d^2)}\right)}{4-d^2} \\ &= \frac{1}{4-d^2}\,\frac{4\left(8-3d^2\right) - d\left(8-2d^2-d^3\right)}{2\left(8-3d^2\right)} \\ &= \frac{32 - 8d - 12d^2 + 2d^3 + d^4}{2\left(4-d^2\right)\left(8-3d^2\right)} \end{aligned}$$

and

$$\begin{aligned} q_2^* &= \frac{2\,(1-w^*) - d}{4-d^2} = \frac{2\left(\frac{8-2d^2-d^3}{2(8-3d^2)}\right) - d}{4-d^2} \\ &= \frac{1}{4-d^2}\,\frac{2\left(8-2d^2-d^3\right) - 2d\left(8-3d^2\right)}{2\left(8-3d^2\right)} \\ &= \frac{16 - 16d - 4d^2 + 4d^3}{2\left(4-d^2\right)\left(8-3d^2\right)}. \end{aligned}$$

This gives us the solution to the (b) part.[3] One can check that, for $d = 0$, the solutions are identical to the ones in the (a) part, as they should be.

Part (c)

For the case $d = 0$, the demands for the two goods are independent of each other, so each firm is a monopolist in its market. This means that we have two vertical relationships with a monopoly both upstream and downstream. That situation is exactly like the simple standard model that we studied in the course (but here we have two parallel such relationships). In such an environment, the effect of integration is that the *double marginalization problem* is avoided. Therefore we should, for the case $d = 0$, expect total surplus to be largest under integration.

- The double marginalization problem: The actions taken by the non-integrated downstream

[2] The effect on the profit that goes through $\widehat{q}_1$ is zero, due to the envelope theorem.

[3] By factoring the numerators, one can simplify these expressions:

$$q_1^* = \frac{(4+d)(2+d)(2-d)^2}{2\left(4-d^2\right)\left(8-3d^2\right)} = \frac{(4+d)(2-d)}{2\left(8-3d^2\right)}$$

and

$$q_2^* = \frac{4\,(1-d)(2-d)(2+d)}{2\left(4-d^2\right)\left(8-3d^2\right)} = \frac{2\,(1-d)}{8-3d^2}.$$

However, this is hard to see and the students are not required to write the solutions in this particular way.

firm influences also the upstream firm's profits. Moreover, internalizing those external effects (which the firms would do after integration) helps also the consumers, not only the upstream firm's profits. In particular, the integrated firm will have a stronger incentive to lower the price, since both the downstream and upstream profits are positively affected by that. Also, a lower price helps consumers and the consumer surplus.

For the case $d = 1$, the two downstream firms compete with each other. Integration between the upstream firm and downstream firm 1 should therefore create an incentive for them to, by rasing w, make it hard for downstream firm 2 to compete. Indeed, the analysis in the (b) part showed that at the equilibrium and for $d = 1$, w will so high that downstream firm 2 leaves the market. Thus, in this case integration has two consequences: (i) as before, the double marginalization effect is avoided and (ii) there is a a negative effect on the extent of downstream competition. Effect (ii) two should, all else equal, have a negative impact on total surplus, whereas effect (i) should again have a positive impact. A priori, it does not seem to be clear which effect is the strongest.

Question 2: Subsidizing a Monopoly Firm's Sales

To the external examiner: The students had seen this model before. It was part of a problem set that was discussed in an exercise class.

Part (a)

The firm's problem: maximize its profits

$$\pi = (a - q)\, q - cq + sq = (a - c + s - q)\, q \qquad (8)$$

with respect to q, subject to $q \geq 0$. Standard calculations yield

$$q^* = \frac{a - c + s}{2}. \qquad (9)$$

This is positive, so the non-negativity constraint does not bind.

Part (b)

We can solve for the subgame-perfect Nash equilibrium by backward induction. We effectively solved the second-stage game in part (a). Plugging (9) into (8) yields

$$\pi^* = (q^*)^2 = \frac{(a - c + s)^2}{4}.$$

Moreover, by standard arguments (the students should derive this from the demand function, though) we have that consumer surplus given q^* is

$$CS = \frac{1}{2}\,(q^*)^2 = \frac{(a - c + s)^2}{8}.$$

Therefore, W given q^* equals

$$\begin{aligned} W &= CS + \pi^* - sq^* \\ &= \frac{(a - c + s)^2}{8} + \frac{(a - c + s)^2}{4} - s\left[\frac{a - c + s}{2}\right]. \end{aligned} \qquad (10)$$

The government wants to maximize this expression with respect to s. Solving yields:

$$s^* = a - c.$$

This is positive, so the non-negativity constraint does not bind.

- Extra: note that s^* yields marginal cost pricing:

$$p^* = a - q^* = a - \frac{a - c + s^*}{2} = c.$$

Part (c)

Modifying the expression in (10), we have

$$\begin{aligned} V &= CS + z\pi^* - sq^* \\ &= \frac{(1 + 2z)\,(10 + s)^2}{8} - \frac{4\,(10 + s)\, s}{8} \\ &= \frac{10 + s}{8}\,[(1 + 2z)\,(10 + s) - 4s] \\ &= \frac{(10 + s)\,[(1 + 2z)\,10 - (3 - 2z)\, s]}{8} \end{aligned}$$

The government wants to maximize this expression with respect to s. The first-order condition is

$$[(1 + 2z)\,10 - (3 - 2z)\, s] - (3 - 2z)\,(10 + s) = 0.$$

The second-order condition is $-2\,(3 - 2z)\, z < 0$, which is always satisfied. Solving the first-order condition for s yields

$$s^{**} = \frac{10\,[(1 + 2z) - (3 - 2z)]}{2\,(3 - 2z)} = \frac{2z - 1}{3 - 2z}10.$$

This is positive, so the non-negativity constraint does not bind.

- s^{**} is increasing in z; therefore, we have from (9) that
$$q^{**} = \frac{10 + s^{**}}{2}$$
also in increasing in z. This in turn means that market price, $p^{**} = a - q^{**}$ is decreasing in z.

- The reason for this result:
  - If you care a lot about the firm's profit (full weight, $z = 1$), then the subsidy does not cost you anything: What you pay out comes back to you with full weight, in terms of profits for the firm. Therefore, you want to subsidize a lot in order to correct the monopolist's incentive to produce too little.
  - If the weight z is smaller, you still think the monopolist produces too little. However, now subsidizing is costly, as you do not get back as much as you pay out. Therefore you choose to subsidize less (i.e., s^{**} is increasing in z, which is consistent with the formula above).
  - Given that the subsidy is increasing in z, it is obvious that the market price is decreasing in z.

- The important thing with this question is that the students show that they can understand the logic of a model—that they are not just mechanically solving the first-order conditions etc. without understanding what they are doing.